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Wave tails in Born–Infeld electrodynamics†

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Abstract. Approximate iterative solutions to the Born–Infeld nonlinear electromagnetic equations are developed in flat space–time. Incoming radiation terms, or wave tails, are shown to arise from the iterative correction of initially purely outgoing approximate solutions.

1. Introduction

In a recent paper, Couch *et al.* (1968), using the Newman–Penrose spin-coefficient formalism (Newman and Penrose 1962), found an approximate form for the 'tail' to a sandwich wave of gravitational radiation exploding from a perturbed Schwarzschild source. Part of the tail consisted of an imploding wave, focused on the source, and arising from the mass–radiation interaction. This incoming wave was interpreted as a backscattering, or reflection, of the emitted wave by the curvature of space–time, which may be regarded as a consequence of the nonlinearity of the Einstein field equations.

In the present paper it is shown that a scattered imploding wave may be attributed entirely to nonlinearity, by the demonstration that it occurs in approximate solutions to the nonlinear Born–Infeld electromagnetic equations (Born and Infeld 1934, Rzewuski 1967) in flat space–time. Solutions corresponding to the monopole, dipole and quadrupole solutions of Maxwell's equations are constructed and the dipole and quadrupole solutions are found to possess incoming tails. The origin of these incoming waves is ascribed to the radiation × radiation × radiation interaction. This form of interaction also gives an incoming tail in the gravitational case (Couch *et al.* 1968).

The relevant aspects of the Born–Infeld theory are summarized in § 2 and the Newman–Penrose formalism reviewed in § 3. In § 4 the Born–Infeld field equations are translated into the spin-coefficient formalism and in §§ 5 and 6 iteration procedures are developed for their solution. Some conclusions are presented in § 7.

2. Born–Infeld electrodynamics

This theory was proposed by Born and Infeld in 1934 in an attempt to mitigate difficulties in Maxwell's theory. However, interest in the theory has been limited by its nonlinearity, which makes solution of the field equations difficult, and quantization impossible, with present procedures.

As in Maxwell's theory, the electromagnetic field is described by a 4-potential A_μ and the Lagrangian is a function of the field quantities $F_{\mu\nu}$ only, where‡

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}. \tag{2.1}$$

Gauge invariance is thus retained. In order to ensure relativistic invariance, the

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‡ A comma denotes partial derivative—a semi-colon, covariant derivative. We work in flat space–time only.

Lagrangian may involve $F_{\mu\nu}$ through the two invariants F and G only, where

$$F = \tfrac{1}{2}b^{-2}F_{\mu\nu}F^{\mu\nu} \tag{2.2}$$

$$G = \tfrac{1}{4}b^{-2}F_{\mu\nu}F^{\mu\nu *} = \tfrac{1}{8}b^{-2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}. \tag{2.3}$$

Here b is a constant with the dimensions of field strength and $\epsilon^{\mu\nu\rho\sigma}$ is the alternating pseudo-tensor. Of the Lagrangians which satisfy these invariance requirements and which reduce to the free field (Maxwell) Lagrangian

$$\mathrm{L} = \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{2.4}$$

for fields weak compared with b, Born and Infeld chose

$$\mathrm{L} = \frac{\{1-(1+F-G^2)^{1/2}\}b^2}{4\pi}. \tag{2.5}$$

In the following, however, a Lagrangian of the form

$$\mathrm{L} = \frac{\{1-(1+F)^{1/2}\}b^2}{4\pi} \tag{2.6}$$

will, for the sake of simplicity, be assumed. Here the term proportional to b^{-4} has been omitted. The field equations resulting from the variation of (2.6) are

$$F^{\mu\nu}{}_{,\nu}(1+F)^{-1/2} = \tfrac{1}{2}b^{-2}F^{\mu\nu}F^{\rho\sigma}F_{\rho\sigma,\nu}(1+F)^{-3/2}. \tag{2.7}$$

It is possible to write these in a form reminiscent of Maxwell's equations by introducing the tensor $P_{\mu\nu} = -F_{\mu\nu}(1+F)^{-1/2}$. The field equations can then be expressed $P^{\mu\nu}{}_{,\nu} = 0$. This device has been used extensively in the literature (e.g. Gilbert 1964, Cornish 1962, Dirac 1960) but will not be adopted here. The only exact solution to (2.7) that appears to have been found is static and spherically symmetric; it is given in § 4.

The form of the field equations used here is

$$F^{\mu\nu}{}_{,\nu} = \tfrac{1}{2}b^{-2}F^{\mu\nu}F^{\rho\sigma}F_{\rho\sigma,\nu}(1+F)^{-1}. \tag{2.8}$$

Thus it is assumed that $1+F \neq 0$, which is justified except, perhaps, at very short distances from the source (see Born and Infeld 1934). Now (2.1) implies that

$$F^{*\mu\nu}{}_{,\nu} = 0 \tag{2.9}$$

so that if

$$F^{\mu\nu +} = \tfrac{1}{2}(F^{\mu\nu} + \mathrm{i}F^{*\mu\nu})$$

then after rearranging (2.8) the field equations may be concisely expressed:

$$F^{\mu\nu}{}_{,\nu}{}^{+} = \tfrac{1}{4}b^{-2}F^{\mu\nu}F^{\rho\sigma}F_{\rho\sigma,\nu} - \tfrac{1}{2}FF^{\mu\nu}{}_{,\nu}. \tag{2.10}$$

3. The Newman–Penrose formalism

In Minkowski space a tetrad of null basis vectors l^μ, n^μ, m^μ and $\bar{m}^\mu$ is introduced, where l^μ and n^μ are real, and m^μ, $\bar{m}^\mu$ complex, subject to

$$l^\mu n_\mu = 1, \qquad m^\mu \bar{m}_\mu = -1.$$

All other contractions of two tetrad vectors give zero. The six real components of the electromagnetic field tensor $F_{\mu\nu}$ are replaced by the three complex quantities

$$\begin{aligned}\phi_0 &= F_{\mu\nu}l^\mu m^\nu \\ \phi_1 &= \tfrac{1}{2}F_{\mu\nu}(l^\mu n^\nu + \bar{m}^\mu m^\nu) \\ \phi_2 &= F_{\mu\nu}\bar{m}^\mu n^\nu.\end{aligned} \tag{3.1}$$

The following operators are defined

$$D = l^\mu \frac{\partial}{\partial x^\mu} \qquad \delta = m^\mu \frac{\partial}{\partial x^\mu} \qquad \Delta = n^\mu \frac{\partial}{\partial x^\mu} \tag{3.2}$$

and the following 12 spin coefficients:

$$\begin{array}{lll}\kappa = l_{\mu;\nu}m^\mu l^\nu & \lambda = -n_{\mu;\nu}\bar{m}^\mu\bar{m}^\nu & \beta = \tfrac{1}{2}(l_{\mu;\nu}n^\mu m^\nu - m_{\mu;\nu}\bar{m}^\mu m^\nu) \\ \pi = -n_{\mu;\nu}\bar{m}^\mu l^\nu & \alpha = \tfrac{1}{2}(l_{\mu;\nu}n^\mu\bar{m}^\nu - m_{\mu;\nu}\bar{m}^\mu\bar{m}^\nu) & \nu = -n_{\mu;\nu}\bar{m}^\mu n^\nu \\ \epsilon = \tfrac{1}{2}(l_{\mu;\nu}n^\mu l^\nu - m_{\mu;\nu}\bar{m}^\mu l^\nu) & \sigma = l_{\mu;\nu}m^\mu m^\nu & \gamma = \tfrac{1}{2}(l_{\mu;\nu}n^\mu n^\nu - m_{\mu;\nu}\bar{m}^\mu n^\nu) \\ \rho = l_{\mu;\nu}m^\mu\bar{m}^\nu & \mu = -n_{\mu;\nu}\bar{m}^\mu m^\nu & \tau = l_{\mu;\nu}m^\mu n^\nu.\end{array} \tag{3.3}$$

The Minkowski metric in null polar coordinates is

$$\mathrm{d}s^2 = \mathrm{d}u^2 + 2\mathrm{d}u\,\mathrm{d}r - r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2). \tag{3.4}$$

where the coordinates are $(x^0, x^1, x^2, x^3) \equiv (u, r, \theta, \phi)$, $u = t - r$. The tetrad is adapted to the null coordinate system by choosing l^μ as the outward null vector tangent to the null cone, n^μ is the inward null vector pointing towards the world line of the origin, and m^μ and $\bar{m}^\mu$ are vectors tangent to the 2-sphere defined by constant r and u. With this assignment

$$l^\mu = \delta_1^\mu \qquad n^\mu = \delta_0^\mu - \tfrac{1}{2}\delta_1^\mu \qquad m^\mu = 2^{-1/2}r^{-1}\{\delta_2^\mu + \mathrm{i}(\sin\theta)^{-1}\delta_3^\mu\} \tag{3.5}$$

and the nonzero spin coefficients become

$$\rho = -r^{-1} \qquad \alpha = -2^{-3/2}r^{-1}\cot\theta \qquad \beta = 2^{-3/2}r^{-1}\cot\theta \qquad \mu = -\tfrac{1}{2}r^{-1}. \tag{3.6}$$

An angular operator ð (edth) is introduced by

$$\begin{aligned}\eth\eta &= -(\sin\theta)^s\left(\frac{\partial}{\partial\theta} + \frac{\mathrm{i}}{\sin\theta}\frac{\partial}{\partial\phi}\right)\{(\sin\theta)^{-s}\eta\} \\ \bar{\eth}\eta &= -(\sin\theta)^{-s}\left(\frac{\partial}{\partial\theta} - \frac{\mathrm{i}}{\sin\theta}\frac{\partial}{\partial\phi}\right)\{(\sin\theta)^s\eta\}\end{aligned} \tag{3.7}$$

where s is the spin weight, defined as follows. η has spin weight s if a transformation $m^\mu \to m^{\mu\prime} = \mathrm{e}^{\mathrm{i}\psi}m^\mu$ induces the transformation $\eta \to \eta' = \mathrm{e}^{\mathrm{i}s\psi}\eta$. From (3.1) the spin weights of ϕ_0, ϕ_1, ϕ_2 are 1, 0, -1 respectively. Spin weighted spherical harmonics ${}_sY_{lm}$ are defined by

$${}_sY_{lm} = \begin{cases}\kappa_{-s}(l)\eth^s Y_{lm} & 0 \leqslant s \leqslant l \\ (-1)^s\kappa_s(l)\bar{\eth}^{-s}Y_{lm} & -l \leqslant s \leqslant 0 \\ 0 & |s| > l\end{cases} \tag{3.8}$$

where ${}_0Y_{lm} = Y_{lm}$ are ordinary spherical harmonics and

$$\kappa_s(l) = \{(l+s)!/(l-s)!\}^{1/2}.$$

It follows from (3.8) that

$$\begin{aligned} \eth_s Y_{lm} &= \{(l-s)(l+s+1)\}^{1/2}\ {}_{s+1}Y_{lm} \\ \bar{\eth}_s Y_{lm} &= -\{(l+s)(l-s+1)\}^{1/2}\ {}_{s-1}Y_{lm} \end{aligned} \tag{3.9}$$

and

$$\eth\bar{\eth}_s Y_{lm} = -(l+s)(l-s+1)\ {}_sY_{lm}. \tag{3.10}$$

4. Born–Infeld equations in Newman–Penrose form

Using the formalism of the preceding section, the Born–Infeld field equations corresponding to (2.10) may be written as four equations in ϕ's and spin coefficients. As with (2.10) Maxwell's equations can be recovered in the limit $b^{-1} \to 0$. The Born–Infeld equations become:

$$\begin{aligned} -\frac{\partial}{\partial u}\phi_1 + \tfrac{1}{2}\frac{\partial}{\partial r}\phi_1 + \frac{1}{r}\phi_1 - 2^{-1/2}r^{-1}\eth\phi_2 \\ = \tfrac{1}{2}b^{-2}\{(\phi_1+\bar{\phi}_1)P - 2^{-1/2}r^{-1}\phi_2 Q - 2^{-1/2}r^{-1}\bar{\phi}_2\bar{Q}\} \\ -b^{-2}R\left(-\frac{\partial}{\partial u}(\phi_1+\bar{\phi}_1) + \tfrac{1}{2}\frac{\partial}{\partial r}(\phi_1+\bar{\phi}_1) - 2^{-1/2}r^{-1}(\eth\phi_2 - \bar{\eth}\bar{\phi}_2) + r^{-1}(\bar{\phi}_1+\phi_1)\right) \end{aligned} \tag{4.1}$$

$$\begin{aligned} \frac{\partial}{\partial r}\phi_1 + 2^{-1/2}r^{-1}\bar{\eth}\phi_0 + 2r^{-1}\phi_1 = \tfrac{1}{2}b^{-2}\{-(\phi_1+\bar{\phi}_1)S + 2^{-1/2}r^{-1}(\bar{\phi}_0 Q + \phi_0\bar{Q})\} \\ -b^{-2}R\left(\frac{\partial}{\partial r}(\phi_1+\bar{\phi}_1) + 2^{-1/2}r^{-1}(\partial\bar{\phi}_0 + \bar{\partial}\phi_0) \right. \\ \left. + 2r^{-1}(\phi_1+\bar{\phi}_1)\right) \end{aligned} \tag{4.2}$$

$$\begin{aligned} \frac{\partial}{\partial r}\phi_2 + 2^{-1/2}r^{-1}\bar{\eth}\phi_1 + r^{-1}\phi_2 = -\tfrac{1}{2}b^{-2}\{\phi_2 S - \bar{\phi}_0 P - (\phi_1-\bar{\phi}_1)2^{-1/2}r^{-1}\bar{Q}\} \\ + b^{-2}R\left(-\frac{\partial}{\partial r}\phi_2 + \frac{\partial}{\partial u}\bar{\phi}_0 - \tfrac{1}{2}\frac{\partial}{\partial r}\bar{\phi}_0 - 2^{-1/2}r^{-1}\bar{\eth}(\phi_1-\bar{\phi}_1) \right. \\ \left. - r^{-1}(\phi_2 + \tfrac{1}{2}\bar{\phi}_0)\right) \end{aligned} \tag{4.3}$$

$$\begin{aligned} -\frac{\partial}{\partial u}\phi_0 + \tfrac{1}{2}\frac{\partial}{\partial r}\phi_0 + \tfrac{1}{2}r^{-1}\phi_0 - 2^{-1/2}r^{-1}\eth\phi_1 = -\tfrac{1}{2}b^{-2}\{\bar{\phi}_2 S - \phi_0 P - (\bar{\phi}_1-\phi_1)2^{-1/2}r^{-1}Q\} \\ + b^{-2}R\left(-\frac{\partial}{\partial r}\bar{\phi}_2 + \frac{\partial}{\partial u}\phi_0 - \tfrac{1}{2}\frac{\partial}{\partial r}\phi_0 - 2^{-1/2}r^{-1}\eth(\bar{\phi}_1-\phi_1) - r^{-1}(\bar{\phi}_2+\tfrac{1}{2}\phi_0)\right) \end{aligned} \tag{4.4}$$

where

$$\begin{aligned} R &= \phi_2\phi_0 + \bar{\phi}_2\bar{\phi}_0 - \phi_1{}^2 - \bar{\phi}_1{}^2 \\ P &= -\frac{\partial R}{\partial u} + \tfrac{1}{2}\frac{\partial R}{\partial r} \\ Q &= -\mathscr{D}R \\ \bar{Q} &= -\overline{\mathscr{D}}R \\ S &= -\frac{\partial R}{\partial r}. \end{aligned}$$

$\mathscr{D} = \partial/\partial\theta + (i/\sin\theta)(\partial/\partial\phi)$, and the $\mathscr{D}$ operator is used instead of $\eth$, $\bar{\eth}$ where it proves convenient in later work.

A static, spherically symmetric solution, first given by Born and Infeld (1934) is, in a cartesian coordinate system:

$$F^{i0} = -ex^i r^{-1}(r_0{}^4 + r^4)^{-1/2} \qquad F^{ik} = 0 \qquad i, k = 1,2,3 \tag{4.5}$$

where

$$r_0{}^2 = e/b \qquad r^2 = (x^1)^2 + (x^2)^2 + (x^3)^2 \qquad e = \text{a real constant.}$$

In null polar coordinates the only nonzero components of $F_{\mu\nu}$ are

$$F_{10} = -F_{01} = e(r_0{}^4 + r^4)^{-1/2} \tag{4.6}$$

and in the Newman–Penrose formalism the solution becomes

$$\phi_0 = \phi_2 = 0 \qquad \phi_1 = \tfrac{1}{2}e(r_0{}^4 + r^4)^{-1/2} \tag{4.7}$$

as may be verified by direct substitution in equations (4.1)–(4.4).

It is also possible to obtain exact solutions, independent of b, which satisfy both equations (4.1)–(4.4) and the source-free Maxwell equations: that is, both sides of equations (4.1)–(4.4) are identically zero. Examples of such solutions are:

$$\text{(i)} \quad \phi_0 = \phi_2 = 0 \qquad \phi_1 = iQ/r^2 \tag{4.8}$$

$$\text{(ii)} \quad \phi_0 = \phi_2 = 0 \qquad \phi_1 = (Q \pm iQ)/r^2 \tag{4.9}$$

where Q is a real constant. It is not obvious how to interpret these solutions, but on the basis of Maxwell's theory (4.8) represents a magnetic monopole and (4.9) combinations of magnetic and electric monopoles. Apparently for these cases the two theories have the same solutions.

5. An iterative method

A successive approximation technique is now developed to solve the field equations such that at the nth stage in the iteration the solution is $\underset{n}{\phi_A}$ where

$$\underset{n}{\phi_A} = \underset{n-1}{\phi_A} + \underset{n}{f_A} \qquad A = 0, 1, 2, \qquad n = 1, 2, \ldots .$$

Here $\underset{n}{f_A}$ is the 'correction' to the $(n-1)$th iterate. The method of solution at each stage in the iteration, used in this section, is applicable only when the initial iterate depends on r alone; for initial iterates of u, r, θ-dependence a different method must be used (see § 6).

The initial iterate ($n = 0$) is taken to be the monopole solution of Maxwell's equations

$$\underset{0}{\phi_0} = \underset{0}{\phi_2} = 0 \qquad \underset{0}{\phi_1} = a_0/r^2 \qquad a_0 = \text{(complex) constant.}$$

This is substituted in the right-hand sides of equations (4.1)–(4.4) and the resulting equations solved for ϕ_A equal to the first iterate $\underset{1}{\phi_A}$. The process is repeated by substituting $\underset{1}{\phi_A}$ in the right-hand sides and solving for $\underset{2}{\phi_A}$, and so on. The method of solution used at each stage was first given in Janis and Newman (1965). The equations to solve for $\underset{1}{\phi_A}$ are (omitting the subscript 1):

$$-\frac{\partial}{\partial u}\phi_1+\tfrac{1}{2}\frac{\partial}{\partial r}\phi_1+r^{-1}\phi_1-2^{-1/2}r^{-1}\eth\phi_2 = b^{-2}r^{-7}(a_0+\bar{a}_0)(a_0{}^2+\bar{a}_0{}^2) \quad (5.1)$$

$$\frac{\partial}{\partial r}\phi_1+2^{-1/2}r^{-1}\bar{\eth}\phi_0+2r^{-1}\phi_1 = 2b^{-2}r^{-7}(a_0+\bar{a}_0)(a_0{}^2+\bar{a}_0{}^2) \quad (5.2)$$

$$\frac{\partial}{\partial r}\phi_2+2^{-1/2}r^{-1}\bar{\eth}\phi_1+r^{-1}\phi_2 = 0 \quad (5.3)$$

$$-\frac{\partial}{\partial u}\phi_0+\tfrac{1}{2}\frac{\partial}{\partial r}\phi_0+\tfrac{1}{2}r^{-1}\phi_0-2^{-1/2}r^{-1}\eth\phi_1 = 0. \quad (5.4)$$

Multiplying (5.2) by r^2 and integrating yields

$$\phi_1 = -r^{-2}\int 2^{-1/2}r\bar{\eth}\phi_0\,\mathrm{d}r-\tfrac{1}{2}b^{-2}r^{-6}(a_0+\bar{a}_0)(a_0{}^2+\bar{a}_0{}^2)+r^{-2}\phi_1{}^0.$$

Assuming an expansion of the form

$$\phi_0 = \sum_{n\geqslant 1}\{\phi_0{}^{n-1}(u,\theta,\phi)/r^{n+2}\}$$

ϕ_1 becomes

$$\phi_1 = r^{-2}\phi_1{}^0+2^{-1/2}\bar{\eth}\sum_{n\geqslant 1}\frac{\phi_0{}^{n-1}}{nr^{n+2}}-\tfrac{1}{2}b^{-2}r^{-6}(a_0+\bar{a}_0)(a_0{}^2+\bar{a}_0{}^2). \quad (5.5)$$

Multiplying equation (5.3) by r and integrating,

$$\phi_2 = 2^{-1/2}r^{-2}\bar{\eth}\phi_1{}^0+\tfrac{1}{2}\bar{\eth}^2\sum_{n\geqslant 1}\frac{\phi_0{}^{n-1}}{n(n+1)r^{n+2}}+\phi_2{}^0r^{-1}. \quad (5.6)$$

Substituting (5.5) in (5.4) and equating coefficients of r^{-3} and r^{-n-3} results in

$$\dot{\phi}_0{}^0 = -2^{-1/2}\eth\phi_1{}^0 \qquad \left(\dot{\phi}_0{}^0 \equiv \frac{\partial}{\partial u}\phi_0{}^0\right) \quad (5.7)$$

$$\dot{\phi}_0{}^n = -\tfrac{1}{2}(n+1)\phi_0{}^{n-1}-\tfrac{1}{2}n^{-1}\eth\bar{\eth}\phi_0{}^{n-1}. \quad (5.8)$$

Substituting (5.3) and (5.6) in (5.1) and equating coefficients of r^{-2} yields

$$\dot{\phi}_1{}^0 = -2^{-1/2}\eth\phi_2{}^0. \quad (5.9)$$

Equations (5.7), (5.8) and (5.9) are exactly the same as those that occur in the solution of the source-free Maxwell equations (Janis and Newman 1965), so with the help of results obtained by them, and with the choice $\phi_2{}^0 = 0$, one finds that $\phi_0{}^0$ and $\phi_1{}^0$ are independent of u. To ensure that the first iterate 'contains' the 0th iterate, $\phi_1{}^0$ and $\phi_0{}^0$ are chosen to be a_0 and 0 respectively. The first iterate is now

$$\underset{1}{\phi_0} = \underset{1}{\phi_2} = 0 \qquad \underset{1}{\phi_1} = a_0r^{-2}+\alpha_1r^{-6}$$

where $\alpha_1 = -\tfrac{1}{2}b^{-2}(a_0+\bar{a}_0)(a_0{}^2+\bar{a}_0{}^2)$. Repeating this procedure twice gives

$$\underset{3}{\phi_0} = \underset{3}{\phi_2} = 0 \qquad \underset{3}{\phi_1} = a_0r^{-2}+\alpha_1r^{-6}+\beta_2r^{-10}+\gamma_3r^{-14}+\mathrm{O}(r^{-18})$$

where

$$\beta_2 = -\alpha_1 b^{-2}\{(a_0+\bar{a}_0)^2 - \tfrac{1}{2}(a_0^2+\bar{a}_0^2)\}$$
$$\gamma_3 = -2a_0\bar{a}_0\beta_2 b^{-2} - \alpha_1^2(a_0+\bar{a}_0)b^{-2}.$$

It is apparent that α_1 depends on b^{-2}, β_2 on b^{-4} and γ_3 on b^{-6}, so each successive iterate contains terms of higher order r^{-1} and b^{-1} dependence. In this sense each iterate is 'correct' to the appropriate powers of r^{-1} and b^{-1}.

If a_0 is put equal to $e/2$, e real, in $\underset{3}{\phi}_A$, this solution turns out to be equal to the power series expansion of the exact solution (4.7), to the accuracy of $\underset{3}{\phi}_A$, so for $a_0 = e/2$ it seems likely that the iterative technique used here converges to a known solution.

6. The Born–Infeld tail

For solutions with u, r and θ dependence the equations for each iterate are solved by a method of Torrence and Janis (1967). The successive iteration scheme is as outlined in § 5 except that, for technical reasons, $\underset{n}{\phi}_A$ does not contain $\underset{0}{\phi}_A$. However as the equations for $\underset{n}{\phi}_A$ are linear and $\underset{0}{\phi}_A$ is a solution of the left-hand sides of these equations equated to zero, $\underset{0}{\phi}_A$ can be added to $\underset{n}{\phi}_A$, $n = 1,2,3,\ldots$ to give the complete solution at each stage.

6.1. *Dipole type solution*

The (axisymmetric) Maxwell dipole solution is taken as initial iterate. This is (Janis and Newman 1965)

$$\begin{aligned}
\underset{0}{\phi}_0 &= a_1(u)\sin\theta/r^3 \\
\underset{0}{\phi}_1 &= -2^{1/2}\dot{a}_1\cos\theta/r^2 - 2^{1/2}a_1\cos\theta/r^3 \\
\underset{0}{\phi}_2 &= -\ddot{a}_1\sin\theta/r - \dot{a}_1\sin\theta/r^2 - a_1\sin\theta/r^3.
\end{aligned} \tag{6.1}$$

As in § 5 these values for $\underset{0}{\phi}_A$ are substituted in the right-hand sides of equations (4.2) and (4.3) and on multiplying by r^2 and r respectively and integrating yield

$$\phi_1 = -2^{-1/2}r^{-2}\int r\eth\phi_0\,\mathrm{d}r + \frac{\phi_1{}^0(u,\theta)}{r^2} - b^{-2}\sum_{k=0}^{4}\frac{c_k(u,\theta)}{r^{k+5}} \tag{6.2}$$

$$\phi_2 = -2^{-1/2}r^{-1}\int \eth\phi_1\,\mathrm{d}r + \frac{\phi_2{}^0(u,\theta)}{r} + b^{-2}\sum_{k=1}^{5}\frac{d_k(u,\theta)}{r^{k+4}} \tag{6.3}$$

where c_k and d_k are functions of a_1, $\bar{a}_1$ and their u derivatives to the third, and of θ. They are given explicitly in Appendix 1. To simplify the equations the constants of r integration, $\phi_1{}^0$ and $\phi_2{}^0$ are set equal to 0. By substituting (6.1) in the right-hand sides of (4.1), (4.4) one obtains

$$\frac{\partial}{\partial u}\phi_0 - \tfrac{1}{2}\frac{\partial}{\partial r}\phi_0 - \tfrac{1}{2}r^{-1}\phi_0 + 2^{-1/2}r^{-1}\eth\phi_1 - b^{-2}\sum_{k=1}^{5}\frac{e_k}{r^{k+5}} = 0 \tag{6.4}$$

$$\frac{\partial}{\partial u}\phi_1 - \tfrac{1}{2}\frac{\partial}{\partial r}\phi_1 - r^{-1}\phi_1 + 2^{-1/2}r^{-1}\eth\phi_2 + b^{-2}\sum_{k=1}^{5}\frac{f_k}{r^{k+5}} = 0 \tag{6.5}$$

where e_k, f_k are given in Appendix 1. Substitution of (6.2) in (6.4) and (6.3) in (6.5) yields

$$\left(\frac{\partial}{\partial u}-\tfrac{1}{2}\frac{\partial}{\partial r}\right)\phi_0-\tfrac{1}{2}r^{-1}\phi_0-\tfrac{1}{2}r^{-3}\eth\bar{\eth}\int r\phi_0\,\mathrm{d}r-b^{-2}\sum_{k=1}^{5}\frac{2^{-1/2}\eth c_{k-1}+e_k}{r^{k+5}}=0 \tag{6.6}$$

$$\left(\frac{\partial}{\partial u}-\tfrac{1}{2}\frac{\partial}{\partial r}\right)\phi_1-r^{-1}\phi_1-\tfrac{1}{2}r^{-2}\eth\bar{\eth}\int \phi_1\,\mathrm{d}r+b^{-2}\sum_{k=1}^{5}\frac{2^{-1/2}\bar{\eth}d_k+f_k}{r^{k+5}}=0. \tag{6.7}$$

We may regard c_k, d_k, e_k and f_k as having spin weight 0, -1, 1 and 0 respectively when they occur in an equation containing ϕ_A. Accordingly the last terms of (6.6) and (6.7) can be written in the form

$$2^{-1/2}\eth c_{k-1}+e_k=\sum_{l=1}s_{k,l}(u)\ {}_1Y_{l0}\qquad k=1,\ldots,5 \tag{6.8}$$

$$2^{-1/2}\bar{\eth}d_k+f_k=\sum_{l=1}t_{k,l}(u)\ {}_0Y_{l0}\qquad k=1,\ldots,5 \tag{6.9}$$

(see Appendix 2). Following Torrence and Janis (1967) and Couch *et al.* (1968), ϕ_0 is taken to be of the form

$$\phi_0=\sum_{l=1}g_{0l}\ {}_1Y_{l0}\qquad g_{0l}=r^{l-1}D^{l+1}(x_0r^{-l}). \tag{6.10}$$

The problem now reduces to finding the form of x_0. Using (6.8), (6.10), (3.10), (6.6) becomes

$$\left(\frac{\partial}{\partial u}-\tfrac{1}{2}\frac{\partial}{\partial r}\right)g_{0l}-\tfrac{1}{2}r^{-1}g_{0l}+\tfrac{1}{2}r^{-3}(l+1)l\int_{\infty}^{r}r'g_{0l}\,\mathrm{d}r'=b^{-2}\sum_{k=1}^{5}\frac{s_{k,l}(u)}{r^{k+5}}. \tag{6.11}$$

Now, if the operator acting on g_{0l} is L,

$$\mathrm{L}g_{0l}\equiv r^{l-1}D^{l+1}(\dot{x}_0r^{-l})-\tfrac{1}{2}lr^{l-2}D^{l+1}(x_0r^{-l})-\tfrac{1}{2}r^{l-1}D^{l+1}(r^{-l}Dx_0-r^{-l-1}lx_0)$$
$$+\tfrac{1}{2}r^{-3}(l+1)l\int r^lD^{l+1}(x_0r^{-l})\,\mathrm{d}r. \tag{6.12}$$

There is an identity given in Torrence and Janis (1967)

$$D\{r^{l+1}D^l(Fr^{-1})\}=r^lD^{l+1}F\qquad\text{for all integers } l\geqslant 0 \tag{6.13}$$

and arbitrary F. Using (6.13) in the second and fourth terms of (6.12) and assuming that $r^{l+1}D^l(x_0r^{-l-1})|_{\infty}=0$:

$$\mathrm{L}g_{0l}\equiv r^{l-1}D^{l+1}\{r^{-l}(\dot{x}_0-\tfrac{1}{2}Dx_0)\}.$$

Equation (6.11) now becomes

$$\dot{x}_0-\tfrac{1}{2}Dx_0=\sum_k(-1)^{l+1}b^{-2}r^{l-k-3}s_{k,l}\frac{(k+2)!}{(l+k+3)!}$$

where all constants of r integration have been put equal to zero. The independent variables are now transformed according to:

$$(u,r)\rightarrow(u,v)\qquad v=u+2r.$$

Under this transformation $\partial/\partial u - \frac{1}{2}\partial/\partial r$ becomes $\partial/\partial u$ and

$$x_0 = \int_{-\infty}^{u} \left((-1)^{l+1} b^{-2} \sum_k \frac{2^{k+3-l} s_{k,l}(u')(v-u')^{l-k-3}(k+2)!}{(k+l+3)!}\right) du' + H_0(v) \qquad (6.14)$$

where H_0 is a constant of u integration. So from (6.10)

$$\phi_0 = \sum_l {}_1Y_{l0}\, r^{l-1} D^{l+1}(x_0 r^{-l}) \qquad (6.15)$$

with x_0 given by (6.14). The equation for ϕ_1, (6.7), is treated similarly, assuming $r^l D^{l-1}(x_1 r^{-l-2})|_\infty = 0$, and

$$\phi_1 = \sum_l {}_0Y_{l0}\, r^{l-1} D^l(x_1 r^{-l-1}) \qquad (6.16)$$

where

$$x_1 = \int_{-\infty}^{u} \left((-1)^{l+1} b^{-2} \sum_k \frac{2^{k-l+3} t_{k,l}(u')(v-u')^{l-k-3}(k+3)!}{(k+l+3)!}\right) du' + H_1(v). \qquad (6.17)$$

$H_1(v)$ is a constant of u integration. ϕ_2 is obtained from equation (6.3), and, using (6.16) and (6.13),

$$\phi_2 = \sum_l {}_{-1}Y_{l0}[\{\tfrac{1}{2}(l+1)l\}^{1/2} r^{l-1} D^{l-1}(x_1 r^{-l-2})] + b^{-2} \sum_{k=1}^{5} \frac{d_k}{r^{k+4}}. \qquad (6.18)$$

The complete solution for the first iterate is obtained when $\underset{0}{\phi_A}$ is added to the results just obtained, so referring to Appendix 2,

$$\begin{aligned}
\phi_0 &= (\tfrac{8}{3}\pi)^{1/2} a_1 r^{-3}\, {}_1Y_{10} + D^2(x_{0,1} r^{-1})\, {}_1Y_{10} + r^2 D^4(x_{0,3} r^{-3})\, {}_1Y_{30} \qquad (6.19)\\
\phi_1 &= \tfrac{1}{2}(\tfrac{8}{3}\pi)^{1/2} d(a_1 r^{-2})\, {}_0Y_{10} + D(x_{1,1} r^{-2})\, {}_0Y_{10} + r^2 D^3(x_{1,3} r^{-4})\, {}_0Y_{30}\\
\phi_2 &= \tfrac{1}{4}(\tfrac{8}{3}\pi)^{1/2} d^2(a_1 r^{-1})\, {}_{-1}Y_{10} + x_{1,1} r^{-3}\, {}_{-1}Y_{10} + 6^{1/2} r^2 D^2(x_{1,3} r^{-5})\, {}_{-1}Y_{30}\\
&\quad + b^{-2} \sum_{k=1}^{5} \frac{d_k}{r^{k+4}}
\end{aligned}$$

where $d = -2\partial/\partial u + \partial/\partial r$ and the constants H_0 and H_1 representing arbitrary incoming radiation have been put equal to zero. $x_{0,1}$ means x_0 for $l = 1$.

An interpretation of these results can be obtained if $s_{k,1}$, $s_{k,3}$, $t_{k,1}$ and $t_{k,3}$ are arranged, by choice of a_1, to be equal to the Dirac delta function or its derivative. In this case the x_0 and x_1 terms reduce to functions of $v = u+2r$ only if $u > 0$ and zero if $u < 0$. Now comparing with the retarded and advanced 2^l pole Maxwell fields, which are respectively given by

$$\phi_A = n_A \mathrm{d}^{l-1+A}\{a_l(u) r^{A-l-2}\} r^{l-1}\, {}_{1-A}Y_{l0}$$
$$\phi_A = m_A D^{l+1-A}\{b_l(u+2r) r^{-A-l}\} r^{l-1}\, {}_{1-A}Y_{l0}$$

where n_A and m_A are numerical factors. The 'correction' terms to the (outgoing) 0th iterate represent incoming radiation, apart from the last term in ϕ_2 which is additional outgoing radiation.

The problem now is to choose the form of a_1. It cannot be put equal to $\delta(u)$ directly, as the $s_{k,l}$ and $t_{k,l}$ terms involve products of the a_1 and their derivatives.

Instead a_1 is put equal to $\epsilon/(u^2+\epsilon^2)$ where ϵ is a positive constant, and if, after evaluating $s_{k,l}$ and $t_{k,l}$, ϵ is made to approach zero, then $s_{k,l}$ and $t_{k,l}$ approximate to $\delta(u)$ or $\partial\{\delta(u)\}/\partial u$.

6.2. *Quadrupole type solution*

The procedure given in § 6.1 can be used with the Maxwell quadrupole solution as initial iterate, and results of the same kind are obtained. The detailed calculations, which are not given here, are, of course, more complicated than those for the dipole case.

7. Conclusions

Tails or backscattered radiation occur in approximate solutions of the Born–Infeld field equations obtained by iteration from the Maxwell dipole and quadrupole solutions. There is no radiation in solutions obtained from the Maxwell monopole solution. The form of the first iterates in the dipole and quadrupole type solutions is:

(i) The initial or 0th iterate in the form of a delta function pulse of radiation outgoing from the source. (The first terms of (6.19).)

(ii) A tail, or backscattered radiation, focused on the source and emanating from (i). (The second and third terms of (6.19).)

(iii) Additional outgoing radiation coincident with (i). (The last term in ϕ_2 in (6.19).)

The radiation described by (ii) and (iii) is of order b^{-2}. Subsequent iterates may be expected to be of higher order b^{-1} dependence. The results are shown in pictorial form by a Penrose picture (figure 1, Penrose 1964).

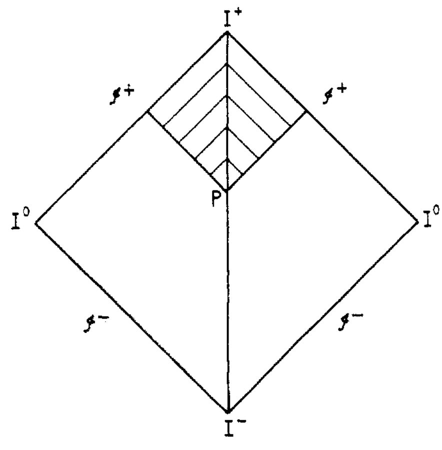

Figure 1. $\mathscr{I}^+$, $\mathscr{I}^-$ denote future and past null infinity, I^+, I^- future and past temporal infinity and I^0 spatial infinity. The line joining I^+ and I^- represents the world line of a source which emits a delta function pulse of radiation at P. The backscattered radiation emanates from this pulse and is focused on the source.

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Appendix 1

$$c_0 = 0$$

$$c_1 = -\sqrt{2}(\dot{a}_1+\dot{\bar{a}}_1)\cos\theta\{(\dot{a}_1{}^2+\dot{\bar{a}}_1{}^2)\cos^2\theta+\tfrac{1}{2}(a_1\ddot{a}_1+\bar{a}_1\ddot{\bar{a}}_1)\sin^2\theta\}$$

$$\begin{aligned} c_2 = & -\sqrt{2}(\dot{a}_1+\dot{\bar{a}}_1)(a_1\dot{a}_1+\bar{a}_1\dot{\bar{a}}_1)\cos\theta\,(2\cos^2\theta+\tfrac{1}{2}\sin^2\theta)-\tfrac{1}{5}\sqrt{2}(a_1+\bar{a}_1)(\dot{a}_1{}^2+\dot{\bar{a}}_1{}^2) \\ & \times\cos\theta(4\cos^2\theta+\sin^2\theta)-\tfrac{3}{10}\sqrt{2}(a_1+\bar{a}_1)(a_1\ddot{a}_1+\bar{a}_1\ddot{\bar{a}}_1)\sin^2\theta\cos\theta \end{aligned}$$

$$\begin{aligned} c_3 = & -\sqrt{2}(\dot{a}_1+\dot{\bar{a}}_1)(a_1{}^2+\bar{a}_1{}^2)\cos\theta(\cos^2\theta+\tfrac{1}{4}\sin^2\theta)-\tfrac{1}{3}\sqrt{2}(a_1+\bar{a}_1)(a_1\dot{a}_1+\bar{a}_1\dot{\bar{a}}_1) \\ & \times\cos\theta(5\cos^2\theta+2\sin^2\theta) \end{aligned}$$

$$c_4 = -\sqrt{2}(a_1+\bar{a}_1)(a_1{}^2+\bar{a}_1{}^2)\cos\theta(\tfrac{6}{7}\cos^2\theta+\tfrac{9}{28}\sin^2\theta)$$

$$d_1 = \ddot{a}_1\sin\theta\{(\dot{a}_1{}^2+\dot{\bar{a}}_1{}^2)\cos^2\theta+\tfrac{1}{2}(a_1\ddot{a}_1+\bar{a}_1\ddot{\bar{a}}_1)\sin^2\theta\}$$

$$\begin{aligned} d_2 = & \ddot{a}_1(a_1\dot{a}_1+\bar{a}_1\dot{\bar{a}}_1)\sin\theta(2\cos^2\theta+\tfrac{1}{2}\sin^2\theta)+\tfrac{2}{5}(\dot{a}_1+\dot{\bar{a}}_1)(\dot{a}_1{}^2+\dot{\bar{a}}_1{}^2)\cos^2\theta\sin\theta \\ & +\tfrac{1}{5}(a_1\ddot{a}_1+\bar{a}_1\ddot{\bar{a}}_1)\sin\theta\{2\dot{a}_1{}^2\sin^2\theta-(\dot{\bar{a}}_1-\dot{a}_1)\cos^2\theta\} \\ & +\tfrac{1}{5}(\dot{a}_1\ddot{a}_1+\dot{\bar{a}}_1\ddot{\bar{a}}_1)\sin\theta(-2\bar{a}_1\cos^2\theta-\tfrac{1}{2}\bar{a}_1\sin^2\theta)-\tfrac{1}{10}\bar{a}_1\sin^3\theta(a_1\dddot{a}_1+\bar{a}_1\dddot{\bar{a}}_1) \end{aligned}$$

$$\begin{aligned} d_3 = & \ddot{a}_1(a_1{}^2+\bar{a}_1{}^2)\sin\theta(\cos^2\theta+\tfrac{1}{4}\sin^2\theta)+\tfrac{1}{6}(a_1\dot{a}_1+\bar{a}_1\dot{\bar{a}}_1)\sin\theta \\ & \times\{(3\dot{\bar{a}}_1+7\dot{a}_1)\cos^2\theta+\tfrac{5}{2}\dot{a}_1\sin^2\theta\}-\tfrac{1}{3}\bar{a}_1(\dot{a}_1{}^2+\dot{\bar{a}}_1{}^2)\sin\theta(\cos^2\theta+\tfrac{1}{4}\sin^2\theta) \\ & +\tfrac{1}{6}(a_1\ddot{a}_1+\bar{a}_1\ddot{\bar{a}}_1)\sin\theta\{(a_1-3\bar{a}_1)\cos^2\theta+(a_1-\tfrac{3}{2}\bar{a}_1)\sin^2\theta\} \end{aligned}$$

$$\begin{aligned} d_4 = & \tfrac{3}{14}(a_1{}^2+\bar{a}_1{}^2)\sin\theta\{(3\dot{a}_1+\dot{\bar{a}}_1)\cos^2\theta+\dot{a}_1\sin^2\theta\}+\tfrac{2}{7}(a_1\dot{a}_1+\bar{a}_1\dot{\bar{a}}_1)\sin\theta \\ & \times\{(a_1-2\bar{a}_1)\cos^2\theta+\tfrac{1}{8}(5a_1-7\bar{a}_1)\sin^2\theta\} \end{aligned}$$

$$d_5 = \tfrac{3}{16}(a_1{}^2+\bar{a}_1{}^2)(a_1-\bar{a}_1)\sin\theta(\cos^2\theta+\tfrac{1}{2}\sin^2\theta)$$

$$e_1 = 2\ddot{\bar{a}}_1\sin\theta\{2(\dot{a}_1{}^2+\dot{\bar{a}}_1{}^2)\cos^2\theta+(a_1\ddot{a}_1+\bar{a}_1\ddot{\bar{a}}_1)\sin^2\theta\}$$

$$\begin{aligned} e_2 = & 5\ddot{\bar{a}}_1(a_1\dot{a}_1+\bar{a}_1\dot{\bar{a}}_1)\sin\theta(2\cos^2\theta+\tfrac{1}{2}\sin^2\theta)+4\dot{a}_1(\dot{a}_1{}^2+\dot{\bar{a}}_1{}^2)\sin\theta\cos^2\theta \\ & -\tfrac{1}{2}a_1(a_1\dddot{a}_1+\bar{a}_1\dddot{\bar{a}}_1)\sin^3\theta+2(a_1\ddot{a}_1+\bar{a}_1\ddot{\bar{a}}_1)\sin\theta\{(2\dot{\bar{a}}_1-\dot{a}_1)\cos^2\theta+\dot{\bar{a}}_1\sin^2\theta\} \\ & -a_1(\dot{a}_1\ddot{a}_1+\dot{\bar{a}}_1\ddot{\bar{a}}_1)\sin\theta(2\cos^2\theta+\tfrac{1}{2}\sin^2\theta) \end{aligned}$$

$$\begin{aligned} e_3 = & 3\ddot{\bar{a}}_1(a_1{}^2+\bar{a}_1{}^2)\sin\theta(2\cos^2\theta+\tfrac{1}{2}\sin^2\theta)+(a_1\dot{a}_1+\bar{a}_1\dot{\bar{a}}_1)\sin\theta \\ & \times\{2(2\dot{\bar{a}}_1+3\dot{a}_1)\cos^2\theta+\tfrac{5}{2}\dot{\bar{a}}_1\sin^2\theta\}-(\dot{a}_1{}^2+\dot{\bar{a}}_1{}^2)\sin\theta(2\bar{a}_1\cos^2\theta+\tfrac{1}{2}a_1\sin^2\theta) \\ & +(a_1\ddot{a}_1+\bar{a}_1\ddot{\bar{a}}_1)\sin\theta\{2(\bar{a}_1-2a_1)\cos^2\theta+(\bar{a}_1-\tfrac{3}{2}a_1)\sin^2\theta\} \end{aligned}$$

$$\begin{aligned} e_4 = & 3(a_1{}^2+\bar{a}_1{}^2)\sin\theta\{(\dot{\bar{a}}_1+\dot{a}_1)\cos^2\theta+\tfrac{1}{2}\dot{\bar{a}}_1\sin^2\theta\} \\ & -(a_1\dot{a}_1+\bar{a}_1\dot{\bar{a}}_1)\sin\theta\{(a_1+\bar{a}_1)\cos^2\theta-\tfrac{1}{4}(5\bar{a}_1-7a_1)\sin^2\theta\} \end{aligned}$$

$$e_5 = \tfrac{3}{4}(a_1{}^2+\bar{a}_1{}^2)(\bar{a}_1-a_1)\sin^3\theta$$

$$\begin{aligned} f_1 = & -\sqrt{2}(\dot{a}_1+\dot{\bar{a}}_1)(\dot{a}_1\ddot{a}_1+\dot{\bar{a}}_1\ddot{\bar{a}}_1)\cos\theta(2\cos^2\theta+\tfrac{1}{2}\sin^2\theta)-\tfrac{1}{2}\sqrt{2}(\dot{a}_1+\dot{\bar{a}}_1)(a_1\dddot{a}_1+\bar{a}_1\dddot{\bar{a}}_1) \\ & \times\sin^2\theta\cos\theta+\sqrt{2}(\ddot{a}_1+\ddot{\bar{a}}_1)\sin^2\theta\cos\theta\{\tfrac{1}{2}(a_1\ddot{a}_1+\bar{a}_1\ddot{\bar{a}}_1)-(\dot{a}_1{}^2+\dot{\bar{a}}_1{}^2)\} \end{aligned}$$

$$\begin{aligned} f_2 = & -\sqrt{2}(\dot{a}_1+\dot{\bar{a}}_1)(\dot{\bar{a}}_1{}^2+\dot{a}_1{}^2)\cos\theta(4\cos^2\theta+\tfrac{3}{2}\sin^2\theta)-\sqrt{2}(\dot{a}_1+\bar{a}_1)(a_1\ddot{a}_1+\bar{a}_1\ddot{\bar{a}}_1) \\ & \times(2\cos^2\theta+\sin^2\theta)-\sqrt{2}(a_1+\bar{a}_1)(\dot{a}_1\ddot{a}_1+\dot{\bar{a}}_1\ddot{\bar{a}}_1)\cos\theta(2\cos^2\theta+\tfrac{1}{2}\sin^2\theta) \\ & -\tfrac{1}{2}\sqrt{2}\sin^2\theta\cos\theta\{(a_1+\bar{a}_1)(a_1\dddot{a}_1+\bar{a}_1\dddot{\bar{a}}_1)+3(\ddot{a}_1+\ddot{\bar{a}}_1)(a_1\dot{a}_1+\bar{a}_1\dot{\bar{a}}_1)\} \end{aligned}$$

$$
\begin{aligned}
f_3 = & -\sqrt{2}(\dot{a}_1+\dot{\bar{a}}_1)(a_1\dot{a}_1+\bar{a}_1\dot{\bar{a}}_1)\cos\theta(7\cos^2\theta+\tfrac{13}{4}\sin^2\theta)-\sqrt{2}(a_1+\bar{a}_1)(\dot{a}_1{}^2+\dot{\bar{a}}_1{}^2) \\
& \times\cos\theta(4\cos^2\theta+\sin^2\theta)-\sqrt{2}(a_1+\bar{a}_1)(a_1\ddot{a}_1+\bar{a}_1\ddot{\bar{a}}_1)\cos\theta(2\cos^2\theta+\tfrac{5}{4}\sin^2\theta) \\
& -\sqrt{2}\cos\theta\sin^2\theta\{(a_1+\bar{a}_1)(\dot{a}_1{}^2+\dot{\bar{a}}_1{}^2)+\tfrac{3}{4}(\ddot{a}_1+\ddot{\bar{a}}_1)(a_1{}^2+\bar{a}_1{}^2)\} \\
f_4 = & -3\sqrt{2}(\dot{a}_1+\dot{\bar{a}}_1)(a_1{}^2+\bar{a}_1{}^2)\cos\theta(\cos^2\theta+\tfrac{1}{2}\sin^2\theta) \\
& -\sqrt{2}(a_1+\bar{a}_1)(a_1\dot{a}_1+\bar{a}_1\dot{\bar{a}}_1)(7\cos^2\theta+\tfrac{5}{2}\sin^2\theta) \\
f_5 = & -3\sqrt{2}(a_1+\bar{a}_1)(a_1{}^2+\bar{a}_1{}^2)\cos\theta(\cos^2\theta+\tfrac{3}{8}\sin^2\theta).
\end{aligned}
$$

Appendix 2

Equation (6.8) is

$$2^{-1/2}\eth c_{k-1}+e_k = \sum_l s_{k,l}\ {}_1Y_{l0} \qquad k = 1, \ldots, 5.$$

Equation (6.9) is

$$2^{-1/2}\eth d_k+f_k = \sum_l t_{k,l}\ {}_0Y_{l0} \qquad k = 1, \ldots, 5.$$

(i) *To justify* (6.8) c_{k-1} can be expressed:

$$c_{k-1} = A_1(u)\cos\theta+A_2(u)\cos^3\theta \qquad \text{some } A_1, A_2.$$

So

$$\eth c_{k-1} = B_1(u)\sin\theta+B_2(u)\sin^3\theta \qquad \text{some } B_1, B_2.$$

Also

$$e_k = A_3(u)\sin\theta+A_4(u)\sin^3\theta \qquad \text{some } A_3, A_4.$$

Now, using (3.8),

$$
\begin{aligned}
{}_1Y_{10} &= (\tfrac{3}{8\pi})^{1/2}\sin\theta \\
{}_1Y_{20} &= \tfrac{3}{2}(\tfrac{5}{6\pi})^{1/2}\cos\theta\sin\theta \\
{}_1Y_{30} &= \tfrac{3}{8}(\tfrac{7}{3\pi})^{1/2}(4\sin\theta-5\sin^3\theta)
\end{aligned}
$$

$$\sum_l s_{k,l}\ {}_1Y_{l0} = s_{k,1}\sin\theta+s_{k,2}\cos\theta\sin\theta+s_{k,3}(4\sin\theta-5\sin^3\theta)+\ldots .$$

Choosing $s_{k,1}$, $s_{k,3}$ nonzero and all other $s_{k,l} = 0$ this can be expressed:

$$\sum_l s_{k,l}\,{}_1Y_{l0} = \sin\theta(s_{k,1}+4s_{k,3})+\sin^3\theta(-5s_{k,3}).$$

So by choosing $s_{k,l}$ appropriately, equation (6.8) can be satisfied.

(ii) *To justify* (6.9)

$$
\begin{aligned}
d_k &= A_5(u)\sin\theta+A_6(u)\sin^3\theta && \text{some } A_5, A_6 \\
\eth d_k &= B_3(u)\cos\theta+B_4(u)\cos^3\theta && \text{some } B_3, B_4 \\
f_k &= A_7(u)\cos\theta+A_8(u)\cos^3\theta && \text{some } A_7, A_8.
\end{aligned}
$$

Now

$$\sum_l t_{k,l}\ {}_0Y_{l0} = t_{k,1}\cos\theta+t_{k,2}(3\cos^2\theta-1)+t_{k,3}(5\cos{}^3\theta-\cos\theta)+\ldots .$$

Choosing $t_{k,1}$, $t_{k,3}$ nonzero and all other $t_{k,l} = 0$ this may be expressed:

$$\sum_l t_{k,l}\ {}_0Y_{l0} = \cos\theta(t_{k,1}-3t_{k,3})+\cos^3\theta(5t_{k,3}).$$

So by choosing $t_{k,l}$ appropriately equation (6.9) can be satisfied.

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